J. Fluid Mech. (1975), *vol.* 69, *part* 1, *pp.* 191–195
Printed in Great Britain

Higher-order approximation for free shear layers in almost rigid rotations

By L. PAMELA COOK AND G. S. S. LUDFORD

Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, New York 14850

(Received 24 August 1973 and in revised form 7 May 1974)

The free shear layer stemming from a discontinuity in angular velocity at either of two parallel disks in almost rigid rotation with fluid between is re-examined. The sole discrepancy between theory and experiment is unaffected by higher-order approximation unless curvature effects are included, when it is reduced.

1. Introduction

The flow between two infinite disks rotating at slightly different rates was treated by Stewartson (1957), who, in particular, effectively gave the first two leading terms in the free shear layers which originate from discontinuities in angular velocity at either disk. The result was later checked experimentally by Baker (1967), using disks whose (equal) inner portions rotated rigidly at slightly different rates from their outer portions. For the antisymmetric part corresponding to inner portions rotating at equal but opposite differential rates to the outer portions, theory was found to differ from experiment by about 40%. [There was satisfactory agreement for the symmetric part as well as on various other questions.]

In calculating Stewartson's values for his apparatus, Baker made an error, whose correction decreases the discrepancy slightly. The possibility of decreasing it further without taking the curvature of the layer into account is offered by a rough estimate of later terms in the expansion, which Baker must presumably have thought were mainly responsible. But, surprisingly enough, closer inspection reveals that these are of alternating sign and tend to cancel, so that only marginal improvement is obtained from four later terms (which are shown below to give the same accuracy as the complete expansion).

We therefore turn to the neglect of curvature, which, though it is justified for the first two terms to which Stewartson limits himself, is not justified beyond them. Baker left the impression that curvature and nonlinearity together were not responsible for any substantial difference; in fact we find that curvature produces a 5% change. Including the effect of curvature decreases the discrepancy between theory and experiment by 7%.

The object of the present note, then, is to give the predictions of the complete linear theory where there was substantial disagreement with the theory used by Baker, and to show that the improvement is not due to higher-order approximation *per se*, but to the inclusion of curvature in it.

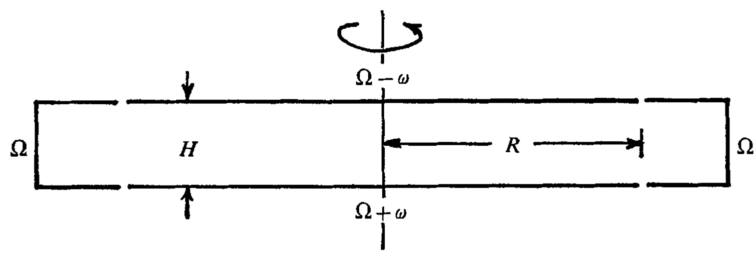

FIGURE 1. Schematic cross-section of Baker's apparatus.

In all other aspects of Baker's experiment there was good agreement with the approximate theory, so that the question arises as to whether this is now disrupted by the complete theory. We checked this and found virtually no change; in fact, the reason can be pinpointed in the analysis. The details are quite uninteresting so we shall not record them here.

2. Analysis

We start by giving a brief but complete analysis of the antisymmetric part tailored to Baker's notation, so as to have the various approximations clearly set out.

The relevant equations (cf. Stewartson) are

$$ED^2 v = \psi_z, \quad ED^4\psi = -v_z,$$

where $\quad E = \dfrac{\nu}{2\Omega H^2}$ (Ekman number), $\quad D^2 = \dfrac{\partial}{\partial r}\dfrac{1}{r}\dfrac{\partial}{\partial r}r + \dfrac{\partial^2}{\partial z^2},$

and the r and z velocity components are

$$u = \psi_z, \quad w = -r^{-1}(r\psi)_r.$$

Here the units of length and velocity are Baker's, namely the distance H between the disks and the velocity differential ωR at the discontinuities (see figure 1, where the sign of ω has been changed to ensure positive w). The boundary conditions become

$$v(r, \pm\tfrac{1}{2}) = \begin{cases} \mp r/R & \text{for} \quad r < R, \\ 0 & \text{for} \quad r > R, \end{cases} \qquad \psi(r, \pm\tfrac{1}{2}) = \psi_z(r, \pm\tfrac{1}{2}) = 0$$

plus the vanishing of velocity on the side wall of the cylinder (which is irrelevant to our analysis).

Taking Hankel transforms shows that

$$\left.\begin{aligned} v(r,z) &= \sum_{i=1}^{3}\int_0^\infty A_i(\xi,E)\,\xi J_1(\xi r)\sinh\alpha_i z\,d\xi, \\ \psi(r,z) &= E\sum_{i=1}^{3}\int_0^\infty \alpha_i^{-1}(-\xi^2+\alpha_i^2)A_i(\xi,E)\,\xi J_1(\xi r)\cosh\alpha_i z\,d\xi, \end{aligned}\right\} \tag{1}$$

where (cf. Stewartson)

$$(\alpha_i^2-\xi^2)^3+\alpha_i^2/E^2 = 0, \quad \sum_{i=1}^{3} A_i \sinh \tfrac{1}{2}\alpha_i = -R\xi^{-1}J_2(\xi R),$$

$$\sum_{i=1}^{3} \alpha_i^{-1}(\alpha_i^2-\xi^2)\, A_i \cosh \tfrac{1}{2}\alpha_i = \sum_{i=1}^{3} (\alpha_i^2-\xi^2) A_i \sinh \tfrac{1}{2}\alpha_i = 0.$$

Everywhere outside the Ekman layers at $z = \pm\frac{1}{2}$ these give

$$A_1 = -E^{\frac{1}{2}}RJ_2(\xi R)C, \quad A_2 = A_3 = 0,$$

to asymptotically exponential accuracy in E, where

$$C = \frac{(1+E^{\frac{2}{3}}\beta^4)^{\frac{1}{2}}}{E^{\frac{1}{6}}\beta(1-E^{\frac{4}{3}}\beta^8-E^{\frac{1}{3}}\beta^2\Delta)\sinh\frac{1}{2}\beta^3+[2\Delta^3+E^{\frac{1}{3}}\beta^2(3+2E^{\frac{2}{3}}\beta^4)\Delta^2]^{\frac{1}{2}}\cosh\frac{1}{2}\beta^3} \tag{2'}$$

and $\Delta = (1+E^{\frac{2}{3}}\beta^4)^{\frac{3}{2}}$. Here β $(=\alpha_1^{\frac{1}{3}})$ is the real root of $\xi = E^{-\frac{1}{3}}\beta(1+E^{\frac{2}{3}}\beta^4)^{\frac{1}{2}}$ which has the same sign as ξ. In arriving at this result we have ignored the side wall of the cylinder, since it provides asymptotically exponentially small corrections to the above and so may be ignored at this stage. Stewartson did not actually write down C but went straight to its two-term approximation, corresponding to (3″) below.

From (1) we compute (with an error less than any power of E)

$$w = -E^{\frac{1}{2}}\int_0^\infty \xi RJ_2(\xi R)\, CJ_0(\xi r)\,(1+E^{\frac{2}{3}}\beta^4)^{\frac{1}{2}}\cosh\beta^3 z\frac{d\xi}{d\beta}d\beta, \tag{2}$$

where
$$\frac{d\xi}{d\beta} = \frac{1}{E^{\frac{1}{3}}}\left[\frac{1+3E^{\frac{2}{3}}\beta^4}{(1+E^{\frac{2}{3}}\beta^4)^{\frac{1}{2}}}\right],$$

this being the component of velocity measured by Baker. The integration variable has been changed to β in order to clarify later steps. If curvature effects are neglected (i.e. $R\to\infty$) this formula becomes

$$w = \pi^{-1}\int_0^\infty B\cosh\beta^3 z\cos[\xi(R-r)]\,d\beta, \tag{3}$$

where
$$B = E^{\frac{1}{6}}(1+3E^{\frac{2}{3}}\beta^4)C. \tag{3'}$$

The result is correct to $O(E^{\frac{1}{3}})$ since, as Stewartson implied, curvature effects are $O(E^{\frac{1}{2}})$. Consistently then he set

$$B = E^{\frac{1}{6}}[E^{\frac{1}{6}}\beta\sinh\tfrac{1}{2}\beta^3+2^{\frac{1}{2}}\cosh\tfrac{1}{2}\beta^3]^{-1}, \tag{3''}$$

a formula which agrees with (3′) to $O(E^{\frac{1}{3}})$. The result (3″), which is due to Stewartson, is the basis of Baker's analysis (although he derives it differently).

In general, (3) can be expected to give accurate results only for very large values of R. For any other R, curvature effects must be considered.

3. Numerical results

In Baker's experiment

$$E = 0{\cdot}0022, \quad R = 12{\cdot}05/1{\cdot}27 \tag{4}$$

in units of $H = 1{\cdot}27$ cm. Numerical integration of (3) with (3″) for B and $r = R$ then gives curve (a) in figure 2 when the unit of velocity is $0{\cdot}0145 \times 12{\cdot}05$ cm/s

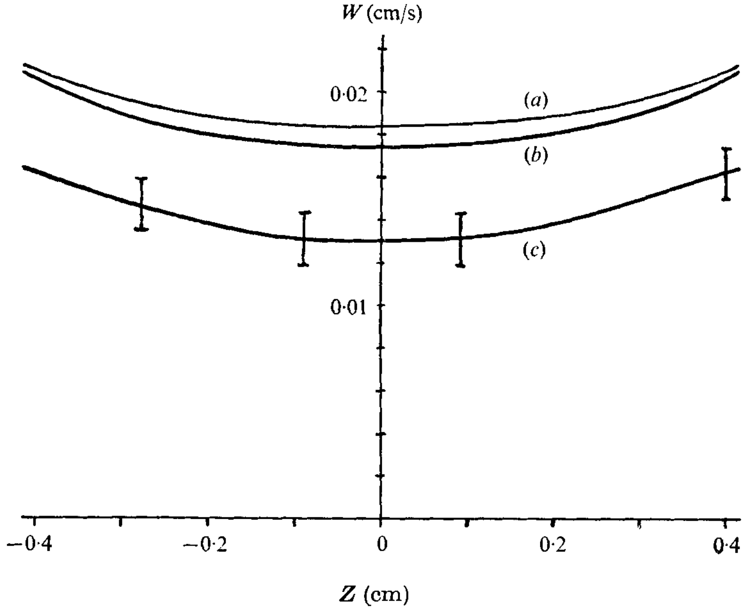

FIGURE 2. Vertical velocity at $r = R$. (*a*) Baker's theoretical result (3) with (3″), recalculated. (*b*) Complete linear result (2) with (2′). (*c*) Baker's measurements. I denotes experimental uncertainty.

and the unit of length is 1·27 cm as in Baker's experiment; W and Z are the dimensional forms of w and z respectively. Baker evaluated the same integral less accurately, but we have not reproduced his curve since the slight difference is not pertinent to the present discussion. Baker's measurements give curve (*c*) and show a discrepancy between theory and experiment of about 40 %.

When the function (3′) is expanded to six terms before integration the first six coefficients $C_i(z)$ are

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix} = \frac{1}{\pi}\int_0^\infty \begin{bmatrix} \frac{1}{2^{\frac{1}{2}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\beta & 0 & 0 & 0 & 0 \\ -3\beta^2/2^{\frac{5}{2}} & 0 & \beta^2/2^{\frac{3}{2}} & 0 & 0 & 0 \\ 0 & 5\beta^3/2^2 & 0 & -\beta^3/2^2 & 0 & 0 \\ 67\beta^4/2^{\frac{11}{2}} & 0 & -17\beta^4/2^{\frac{7}{2}} & 0 & \beta^4/2^{\frac{5}{2}} & 0 \\ 0 & -11\beta^5/2^3 & 0 & \frac{3}{2}\beta^5 & 0 & -\beta^5/2^3 \end{bmatrix} \begin{bmatrix} 1 \\ \tanh\frac{1}{2}\beta^3 \\ \tanh^2\frac{1}{2}\beta^3 \\ \tanh^3\frac{1}{2}\beta^3 \\ \tanh^4\frac{1}{2}\beta^3 \\ \tanh^5\frac{1}{2}\beta^3 \end{bmatrix} \frac{\cosh\beta^3 z}{\cosh\frac{1}{2}\beta^3}\,d\beta.$$

The corresponding terms (in dimensional form) are recorded in table 1 for the value of E given in (4). [Note that it is easier to do a straightforward numerical integration for these coefficients than to sum equivalent residue series, such as that given for C_1 by Stewartson. The same first two coefficients are, of course, obtained by expansion of the function (3″).] The magnitude of all but the last is big enough to give a significant correction to w, but the alternation in their signs results in an insignificant overall one. The corresponding curve is indistinguishable from (*a*) (except near $Z = 0{\cdot}4$, where it is actually higher). Virtually the same

Z (cm)	$E^{\frac{1}{6}}C_1$	$E^{\frac{1}{3}}C_2$	$E^{\frac{1}{2}}C_3$	$E^{\frac{2}{3}}C_4$	$E^{\frac{5}{6}}C_5$	EC_6	Sum of first two columns†	Sum of all six columns $\doteqdot$ integral (3) with (3′)
0	0·01973	−0·00185	−0·00097	0·00093	0·00031	−0·00006	0·0179	0·0181
0·1	0·01989	−0·00191	−0·00098	0·00098	0·00032	−0·00006	0·0180	0·0182
0·2	0·02048	−0·00212	−0·00104	0·00117	0·00034	−0·00006	0·0184	0·0188
0·3	0·02148	−0·00255	−0·00116	0·00156	0·00040	−0·00007	0·0189	0·0197
0·4	0·02337	−0·00338	−0·00139	0·00240	0·00052	−0·00007	0·0200	0·0214

† These values give a curve slightly closer to (*b*) in figure 2 than (*a*), which contains spurious higher-order terms. The six-term sums in the last column actually move the curve away again.

TABLE 1. First six terms for Baker's experiment, neglecting curvature

numbers are obtained by integrating the unexpanded function (3′). We conclude that the discrepancy is not diminished by higher-order terms *per se*.

We now turn to the asymptotically exact formula (2) with C given by (2′). There is no virtue in expanding: we merely note that, if this is done, each of the C_i is altered by the inclusion of the factor $-\pi\xi R J_0(\xi R)\, J_2(\xi R)$ in its integrand and, since $\xi = E^{-\frac{1}{3}}\beta(1+E^{\frac{2}{3}}\beta^4)^{\frac{1}{2}}$, must be expanded out. The values for $R = \infty$ are thereby recovered plus terms of relative order $E^{\frac{1}{3}}$, so that the first term in w affected by the finiteness of R is $O(E^{\frac{1}{2}})$, as Stewartson implied. Numerical integration of (2) now gives curve (*b*) in figure 2, which is about 33% higher than curve (*c*) near the centre. While experimental error does not close the gap (which is still at least 20% when the highest experimental values are used), our results do show that a substantial part of the discrepancy can be attributed to curvature effects.

Assuming that Baker's experiment is reliable and that terms *exponentially* small in E are numerically unimportant, we are forced to attribute the remaining discrepancy to nonlinearity. It is curious that only the 'antisymmetric' problem shows a discrepancy, since the inertia terms are then smaller by a factor $E^{\frac{1}{12}}$ [$=$ 0·6005 for Baker's value of E in (4)] than they are in the 'symmetric' problem. It would be interesting to see whether the azimuthal velocity is affected to the same extent, but unfortunately that was not measured for the antisymmetric problem as it was for the symmetric.

This work was supported in part by the U.S. Army Research Office, Durham, and in part by the National Science Foundation under Grants GP 28483 and 40965.

REFERENCES

BAKER, D. J. 1967 Shear layers in a rotating fluid. *J. Fluid Mech.* **29**, 165–75.
STEWARTSON, K. 1957 On almost rigid rotations. *J. Fluid Mech.* **3**, 17–26.